Exercises

Derivatives – Solutions

Exercise 1.

(a) Let f and g be differentiable. Then with the product rule (P) and the chain rule (C) we get

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\cdot\frac{1}{g(x)}\right)' \\ &= \left(f(x)\cdot(g(x))^{-1}\right)' \\ &\overset{(P)}{=} f'(x)\cdot(g(x))^{-1} + f(x)\cdot\left((g(x))^{-1}\right)' \\ &\overset{(C)}{=} \frac{f'(x)}{g(x)} + f(x)\cdot\left((-1)\cdot(g(x))^{-2}\cdot g'(x)\right) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ &= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

Note: $(g(x))^{-1}$ means the negative power of the value $g(x)$ (i.e. $\frac{1}{g(x)}$) and should not be confused with the inverse g^{-1} of g!

(b) (i) Using the chain rule (C) we get

$$(a^x)' = \left(e^{\ln(a)\cdot x}\right)' \overset{(C)}{=} e^{\ln(a)\cdot x}\cdot\ln(a) = a^x\cdot\ln(a).$$

(ii) Using the chain rule (C) and the product rule (P) we get

$$\begin{aligned}(x^x)' = \left(e^{\ln(x)\cdot x}\right)' &\overset{(C)}{=} e^{\ln(x)\cdot x}\cdot(\ln(x)\cdot x)' \\ &\overset{(P)}{=} e^{\ln(x)\cdot x}\cdot\left(\frac{1}{x}\cdot x + \ln(x)\right) = x^x\cdot(1+\ln(x)).\end{aligned}$$

Exercise 2.

(a) With the chain rule we get

$$f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

(b)

$$f(x) = \frac{1}{x} - x^3 + 2\ln x + e$$

$$\Rightarrow f'(x) = -\frac{1}{x^2} - 3x^2 + \frac{2}{x}$$

(c)

$$f(x) = \sin^2 x \cdot \cos^2 x$$

Product rule: $(u \cdot v)' = u'v + uv'$:

$$\Rightarrow f'(x) = 2\sin x \cdot \cos x \cdot \cos^2 x + \sin^2 x \cdot 2\cos x \cdot (-\sin x)$$

$$= 2\sin x \cos x \,(\cos^2 x - \sin^2 x)$$

(d)

$$f(x) = 2^x + \frac{\ln x}{2} - \frac{1}{x}$$

$$\Rightarrow f'(x) = \ln 2 \cdot 2^x + \frac{1}{2x} + \frac{1}{x^2}$$

(e)

$$f(x) = \frac{x^2}{\sin x + x}$$

Quotient rule:

$$f'(x) = \frac{2x\,(\sin x + x) - x^2(\cos x + 1)}{(\sin x + x)^2} = \frac{2x\sin x + x^2 - x^2\cos x}{(\sin x + x)^2}$$

(f)

$$f(x) = e^{(x+2)^2 - x}$$

Chain rule: $y = e^z,\ z = (x+2)^2 - x$

$$\Rightarrow y'(z) = e^z,\ z'(x) = 2(x+2) - 1 = 2x + 3$$

$$\Rightarrow f'(x) = y'(z) \cdot z'(x) = e^z \cdot (2x+3) = (2x+3)\,e^{(x+2)^2 - x}$$

Exercise 3.

(a) We compute the first and second derivative:

$$f'(x) = 6x^2 + 6x - 36$$
$$f''(x) = 12x + 6$$

We check the necessary condition for extrema and obtain

$$\begin{aligned} & f'(x) = 0 \\ \Leftrightarrow \quad & 6x^2 + 6 - 36 = 0 \\ \Leftrightarrow \quad & x^2 + 1 - 6 = 0 \\ \Leftrightarrow \quad & (x-2)(x+3) = 0 \\ \Leftrightarrow \quad & x = 2 \quad \text{or} \quad x = -3 \end{aligned}$$

So $x_1 = 2$ and $x_2 = -3$ are the possible extreme points. With $f''(x_1) = 12 \cdot 2 + 6 = 30 > 0$ and $f''(x_2) = 12 \cdot (-3) + 6 = -30 < 0$ we know that there is a local minimum at $x_1 = 2$ and a local maximum at $x_2 = -3$.
None of these points is a global extremum since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

(b) We compute the first and second derivative:

$$\begin{aligned} g'(x) &= (2x+3)\, e^{(x+2)^2 - x} \\ g''(x) &= \left((2x+3)\, e^{(x+2)^2 - x}\right)' \\ &= 2e^{(x+2)^2 - x} + (2x+3) \cdot \left((2x+3)\, e^{(x+2)^2 - x}\right) \\ &= \left((2x+3)^2 + 2\right) e^{(x+2)^2 - x} \end{aligned}$$

We check the necessary condition for extrema and obtain

$$\begin{aligned} & g'(x) = 0 \\ \Leftrightarrow \quad & (2x+3)\, \underbrace{e^{(x+2)^2 - x}}_{>0} = 0 \\ \Leftrightarrow \quad & 2x + 3 = 0 \\ \Leftrightarrow \quad & x = \frac{3}{2} \end{aligned}$$

With $g''(x) = \underbrace{\left((2x+3)^2 + 2\right)}_{>0} \underbrace{e^{(x+2)^2 - x}}_{>0} > 0$, we know that there is a local minimum at $x = \frac{3}{2}$.
It is easy to see that $g'(x) < 0$ for $x < \frac{3}{2}$ and $g'(x) > 0$ for $x > \frac{3}{2}$ which means that $g(x)$ is strictly decreasing up to $x = \frac{3}{2}$ and strictly increasing from $x = \frac{3}{2}$. Thus the minimum at $x = \frac{3}{2}$ must be a global minimum.

Exercise 4.

(a)

$$\lim_{x \to 0} \frac{\sin x}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$$

(b)

$$\lim_{x \to 0} \ln(x) \cdot x = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x = 0$$

By transforming the function into $\frac{\ln(x)}{\frac{1}{x}}$ the numerator as well the denominator diverge to $\pm\infty$.

(c)

$$\lim_{x \to 0} \frac{e^x - x - 1 - \frac{1}{2}x^2}{\sin x - x} \overset{L'H}{=} \lim_{x \to 0} \frac{e^x - 1 - x}{\cos x - 1} \overset{L'H}{=} \lim_{x \to 0} \frac{e^x - 1}{-\sin x} \overset{L'H}{=} \lim_{x \to 0} \frac{e^x}{-\cos x} = \frac{1}{-1} = -1$$

Exercise 5.

(a) By the definition of the inverse function, we know that

$$f(f^{-1}(x)) = x.$$

Thus, the functions on both sides are the same and must have the same derivative. This implies

$$\begin{aligned} & \left(f(f^{-1}(x))\right)' = (x)' \\ \Leftrightarrow \quad & f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1 \\ \Leftrightarrow \quad & (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

for all x with $f'(f^{-1}(x)) \neq 0$. This finishes the proof.

(b) We know that $\ln x$ is the inverse of e^x, i.e. for $f(x) := e^x$ we have $f^{-1}(x) = \ln(x)$ and $f'(x) = e^x$. Thus the rule of the derivative of the inverse gives us

$$(\ln(x))' = \left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

(c) (i)

$$
\begin{aligned}
(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1 + (\tan(x))^2.
\end{aligned}
$$

(ii) For $f(x) = \tan(x)$ we have $f^{-1}(x) = \arctan(x)$ and $f'(x) = 1 + \tan^2(x)$. Thus we get

$$
(\arctan(x))' = \left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{1 + (\tan(\arctan(x)))^2} = \frac{1}{1+x^2}.
$$